

# On uniqueness of differential structures on orbifolds

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It is known that every  $C^r$ -orbifold,  $1 \leq r \leq \infty$ , has a compatible  $C^s$ -differential structure, for every  $s$ , where  $r < s \leq \omega$ . We prove that if two reduced  $C^r$ -orbifolds,  $2 \leq r \leq \omega$ , are  $C^2$ -diffeomorphic, then they are  $C^r$ -diffeomorphic. It follows that the compatible  $C^s$ -differential structure on a reduced  $C^r$ -orbifold,  $2 \leq r < s \leq \omega$ , is unique up to a  $C^s$ -diffeomorphism.

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## 1 Introduction

In this note we study differentiable structures on orbifolds. An orbifold is called reduced, if the actions of the local groups on the orbifold charts are effective. Orbifolds that are not reduced can be diffeomorphic even if their orbifold atlases are quite different. Therefore, we only consider reduced orbifolds. We prove the following result:

**Theorem 1.1** *Let  $X$  and  $Y$  be reduced  $C^r$ -differentiable orbifolds,  $2 \leq r \leq \omega$ . If  $X$  and  $Y$  are  $C^2$ -diffeomorphic, then they are  $C^r$ -diffeomorphic.*

As usual,  $C^\infty$  and  $C^\omega$  mean smooth and real analytic, respectively. A  $C^r$ -differentiable structure on an orbifold  $X$  means a maximal  $C^r$ -atlas  $\alpha$  on  $X$ . A  $C^s$ -differentiable structure  $\beta$  on  $X$ ,  $s > r$ , is called *compatible* with  $\alpha$ , if  $\beta \subset \alpha$ . In this case, every chart on  $\beta$  is a chart on  $\alpha$ . For a reduced orbifold  $X$ , this means equivalently that the identity map on  $X$  is a  $C^r$ -orbifold diffeomorphism  $X(\alpha) \rightarrow X(\beta)$ . In [2] (Theorems 7.4 and 8.2), we proved that every  $C^r$ -orbifold  $X$ ,  $1 \leq r \leq \infty$ , has a compatible  $C^s$ -differential structure, for any  $s$  such that  $r < s \leq \omega$ .

Let  $X$  and  $Y$  be reduced orbifolds equipped with  $C^s$ -differential structures  $\alpha$  and  $\beta$ , respectively. Assume there is a  $C^s$ -diffeomorphism  $f: X \rightarrow Y$ . Then  $\alpha$  has a refinement  $\alpha_0$  such that  $f$  takes  $\alpha_0$  to a refinement  $f(\alpha_0)$  of  $\beta$  (Theorem 2.6). Thus the existence result in [2] together with Theorem 1.1 imply the following:

**Theorem 1.2** *Let  $\alpha$  be a  $C^r$ -differential structure on a reduced orbifold  $X$ ,  $2 \leq r \leq \infty$ . There is a  $C^s$ -differentiable structure  $\beta$  on  $X$  compatible with  $\alpha$ , for every  $s$ ,  $r < s \leq \omega$ , and  $\beta$  is unique up to a  $C^s$ -diffeomorphism.*

The proof of Theorem 1.1 is based on using the frame bundle construction for reduced orbifolds: The fact that the general linear group  $GL_n(\mathbb{R})$  acts properly on the frame bundle  $\text{Fr}(X)$  of a reduced  $n$ -dimensional orbifold allows us to use approximation results (see [1]) for differentiable equivariant maps between the frame bundles of two reduced orbifolds.

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## 2 Definitions

We begin with the definition of an orbifold:

**Definition 2.1** Let  $X$  be a topological space and let  $n \in \mathbb{N}$ .

- (1) An *orbifold chart* of  $X$  is a triple  $(\tilde{U}, G, \varphi)$ , where  $\tilde{U}$  is a connected open subset of  $\mathbb{R}^n$ ,  $G$  is a finite group acting on  $\tilde{U}$  and  $\varphi: \tilde{U} \rightarrow X$  is a  $G$ -invariant map that induces a homeomorphism  $U = \varphi(\tilde{U}) \cong \tilde{U}/G$ . The subgroup of  $G$  acting trivially on  $\tilde{U}$  is denoted by  $\ker(G)$ .
- (2) An *embedding*  $(\lambda, \theta): (\tilde{U}, G, \varphi) \rightarrow (\tilde{V}, H, \psi)$  between two orbifold charts is an injective homomorphism  $\theta: G \rightarrow H$  such that  $\theta$  is an isomorphism from  $\ker(G)$  to  $\ker(H)$ , and an equivariant embedding  $\lambda: \tilde{U} \rightarrow \tilde{V}$  with  $\psi \circ \lambda = \varphi$ .
- (3) An *orbifold atlas* on  $X$  is a family  $\mathcal{U} = \{(\tilde{U}, G, \varphi)\}$  of orbifold charts covering  $X$  and satisfying the following: For any two charts  $(\tilde{U}, G, \varphi)$  and  $(\tilde{V}, H, \psi)$  and for any point  $x \in \varphi(\tilde{U}) \cap \psi(\tilde{V})$ , there exists a chart  $(\tilde{W}, K, \mu)$  such that  $x \in \mu(\tilde{W})$  and embeddings  $(\tilde{W}, K, \mu) \rightarrow (\tilde{U}, G, \varphi)$  and  $(\tilde{W}, K, \mu) \rightarrow (\tilde{V}, H, \psi)$ .
- (4) An orbifold atlas  $\mathcal{U}$  *refines* another orbifold atlas  $\mathcal{V}$  if every chart in  $\mathcal{U}$  admits an embedding into some chart in  $\mathcal{V}$ . Two orbifold atlases are called *equivalent* if they have a common refinement.

**Definition 2.2** An  $n$ -dimensional *orbifold* is a paracompact Hausdorff space  $X$  equipped with an equivalence class of  $n$ -dimensional orbifold atlases.

An orbifold is called *reduced* if  $G$  acts effectively on  $\tilde{U}$ , for every orbifold chart  $(\tilde{U}, G, \varphi)$ .

An orbifold is called a  $C^r$ -orbifold,  $1 \leq r \leq \omega$ , where  $C^\infty$  means *smooth* and  $C^\omega$  means *real analytic*, if for every orbifold chart  $(\tilde{U}, G, \varphi)$ , the action of  $G$  on  $\tilde{U}$  is  $C^r$ -differentiable, and if each  $\lambda: \tilde{U} \rightarrow \tilde{V}$  is a  $C^r$ -embedding.

We recall the definition of an orbifold map:

**Definition 2.3** Let  $X$  and  $Y$  be  $C^r$ -orbifolds,  $1 \leq r \leq \omega$ . Let  $0 \leq p \leq r$ . We call a map  $f: X \rightarrow Y$  a  $C^p$ -differentiable orbifold map, if for every  $x \in X$ , there are charts  $(\tilde{U}, G, \varphi)$  around  $x$  and  $(\tilde{V}, H, \psi)$  around  $f(x)$ , such that  $f$  maps  $U = \varphi(\tilde{U})$  into  $V = \psi(\tilde{V})$  and the restriction  $f|_U$  can be lifted to a  $C^p$ -differentiable equivariant map  $\tilde{f}: \tilde{U} \rightarrow \tilde{V}$ .

**Definition 2.4** Let  $X$  and  $Y$  be  $C^r$ -orbifolds,  $1 \leq r \leq \omega$ . Let  $1 \leq p \leq r$ . A map  $f: X \rightarrow Y$  is called a  $C^p$ -diffeomorphism, if  $f$  is a  $C^p$ -differentiable bijection, and if the inverse map  $f^{-1}: Y \rightarrow X$  is  $C^p$ -differentiable. If there is a  $C^p$ -diffeomorphism  $X \rightarrow Y$ , then we call  $X$  and  $Y$   $C^p$ -diffeomorphic.

**Proposition 2.5** Let  $X$  and  $Y$  be reduced  $C^r$ -orbifolds,  $1 \leq r \leq \omega$ . Let  $1 \leq p \leq r$ . Let  $f: X \rightarrow Y$  be a bijection. Then the following are equivalent:

- (1) The bijection  $f$  is a  $C^p$ -diffeomorphism.
- (2) For every  $x \in X$ , there are orbifold charts  $(\tilde{U}, G, \varphi)$  of  $X$  and  $(\tilde{V}, H, \psi)$  of  $Y$  satisfying the following conditions:
  - (a)  $f(U) = V$ , where  $U = \varphi(\tilde{U})$  and  $V = \psi(\tilde{V})$ .
  - (b)  $x = \varphi(\tilde{x})$  and  $f(x) = \psi(\tilde{y})$ , for some  $\tilde{x} \in \tilde{U}$  and  $\tilde{y} \in \tilde{V}$ , respectively.
  - (c)  $G_{\tilde{x}} = G$ ,  $H_{\tilde{y}} = H$ .
  - (d) The restriction  $f|_U: U \rightarrow V$  has an equivariant lift  $\tilde{f}: \tilde{U} \rightarrow \tilde{V}$  that is a  $C^p$ -diffeomorphism and the corresponding homomorphism  $G \rightarrow H$  is an isomorphism.

**Proof** Clearly, a bijection satisfying Conditions 2 (a) - (d) is a  $C^p$ -diffeomorphism. Assume then that  $f: X \rightarrow Y$  is a  $C^p$ -diffeomorphism, and let  $x \in X$ . There are orbifold charts  $(\tilde{U}, G, \varphi)$  and  $(\tilde{U}', G', \varphi')$  of  $X$  around  $x$  and  $(\tilde{W}, H, \psi)$  and  $(\tilde{W}', H', \psi')$  of  $Y$  around  $y = f(x)$ , where  $\varphi(\tilde{U}) = U$ ,  $\varphi'(\tilde{U}') = U'$ ,  $\psi(\tilde{W}) = W$  and  $\psi'(\tilde{W}') = W'$  making the following diagram commute:

$$\begin{array}{ccccccc}
\tilde{U} & \xrightarrow{\tilde{f}} & \tilde{W} & \xrightarrow{\lambda} & \tilde{W}' & \xrightarrow{\tilde{e}} & \tilde{U}' \\
\downarrow \varphi & & \downarrow \psi & & \downarrow \psi' & & \downarrow \varphi' \\
U & \xrightarrow{f} & W & \xrightarrow{i} & W' & \xrightarrow{f^{-1}} & U'
\end{array}$$

Here  $\tilde{f}$  and  $\tilde{e}$  are equivariant  $C^p$ -differentiable lifts of the restrictions  $f|_U$  and  $f^{-1}|_{W'}$ , respectively,  $\lambda: \tilde{W} \rightarrow \tilde{W}'$  is an equivariant  $C^r$ -embedding and  $i: W \rightarrow W'$  is the inclusion. We may choose the orbifold charts in such a way that  $G = G_{\tilde{x}}$ , for some  $\tilde{x} \in \tilde{U}$ , where  $\varphi(\tilde{x}) = x$ , and  $H = H_{\tilde{y}}$ , for some  $\tilde{y} \in \tilde{W}$ , where  $\psi(\tilde{y}) = y$ . Similarly,  $G' = G'_{\tilde{x}'}$  for some  $\tilde{x}' \in \tilde{U}'$ , where  $\varphi'(\tilde{x}') = x$  and  $H' = H'_{\tilde{y}'}$  for some  $\tilde{y}' \in \tilde{W}'$ , where  $\psi'(\tilde{y}') = y$ . Since the local groups are unique up to an isomorphism, it follows that  $G' \cong G$  and  $H' \cong H$ . According to Lemma 2.2 in [5], the composed map  $\tilde{e} \circ \lambda \circ \tilde{f}$  must be an embedding. It follows that  $\tilde{f}$  is an injection and has maximal rank at every point. Therefore,  $\tilde{f}$  is an embedding and  $\tilde{f}(\tilde{U})$  is open in  $\tilde{W}$ .

Let  $\theta: G \rightarrow H$  be the homomorphism associated with the lift  $\tilde{f}$ . Let  $g_1, g_2 \in G$  and assume  $\theta(g_1) = \theta(g_2)$ . Then  $\theta(g_1^{-1}g_2) = e$  and  $\tilde{f}(g_1^{-1}g_2z) = \theta(g_1^{-1}g_2)\tilde{f}(z) = \tilde{f}(z)$ , for all  $z \in \tilde{U}$ . Since  $\tilde{f}$  is an injection, it follows that  $g_1^{-1}g_2z = z$  for all  $z \in \tilde{U}$ . Since  $G$  acts effectively on  $\tilde{U}$ , it follows that  $g_1 = g_2$ . Therefore,  $\theta$  is an injection. Similarly, we can see that the homomorphism associated with the lift  $\tilde{e}$  is injective. By assumption, the homomorphism associated with  $\lambda$  is injective. Since  $G \cong G'$ , it follows that  $\theta$  is an isomorphism. Thus the charts  $(\tilde{U}, G, \varphi)$  and  $(\tilde{V}, H, \psi|)$ , where  $\tilde{V} = \tilde{f}(\tilde{U})$ , satisfy the Conditions 2 (a), (b), (c) and (d).  $\square$

Notice that Proposition 2.5 does not hold without the assumption that  $X$  and  $Y$  are reduced. The remark on p. 2372 in [3] gives an example of an orbifold diffeomorphism that fails to satisfy Condition 2 (d) of Proposition 2.5.

Let  $X, Y$  and  $f$  be as in Proposition 2.5. Denote the  $C^r$ -differential structure on  $X$  by  $\alpha$  and the  $C^r$ -differential structure on  $Y$  by  $\beta$ . Let  $\alpha_0$  be the collection of the charts  $(\tilde{U}, G, \varphi)$  in  $\alpha$  having the property that there is a chart  $(\tilde{V}, H, \psi)$  in  $\beta$ , where  $(\tilde{U}, G, \varphi)$  and  $(\tilde{V}, H, \psi)$  satisfy Conditions 2 (a) - (d) of Proposition 2.5. Thus, for every chart in  $\alpha_0$  we associate a chart in  $\beta$ . We denote by  $f(\alpha_0)$  the collection of charts in  $\beta$  obtained in this way.

**Theorem 2.6** *Let  $X$  and  $Y$  be reduced  $C^r$ -orbifolds,  $1 \leq r \leq \omega$ , and let  $f: X \rightarrow Y$  be a  $C^r$ -diffeomorphism. Let  $\alpha$  and  $\beta$  be the  $C^r$ -differential structures on  $X$  and  $Y$ , respectively. Then:*

- (1) The collection  $\alpha_0$  is a  $C^r$ -atlas on  $X$  refining  $\alpha$ .
- (2) The collection  $f(\alpha_0)$  is a  $C^r$ -atlas on  $Y$  refining  $\beta$ .

**Proof** Let  $(\tilde{U}_i, G_i, \varphi_i) \in \alpha_0$  for  $i = 1, 2$ , and let  $x \in \varphi_1(\tilde{U}_1) \cap \varphi_2(\tilde{U}_2)$ . Let  $(\tilde{V}_i, G_i, \psi_i) \in f(\alpha_0)$ ,  $i = 1, 2$ , be the corresponding charts in  $\beta$ . Let  $\tilde{x}_i \in \tilde{U}_i$  and  $\tilde{y}_i \in \tilde{V}_i$  be such that  $G_i = G_{\tilde{x}_i} = G_{\tilde{y}_i}$ , for  $i = 1, 2$ , where  $\tilde{y}_i = \tilde{f}_i(\tilde{x}_i)$  and  $\tilde{f}_i: \tilde{U}_i \rightarrow \tilde{V}_i$  is an equivariant lift of  $f|: \varphi_i(\tilde{U}_i) \rightarrow \psi_i(\tilde{V}_i)$  as in Condition 2 (d) of Proposition 2.5.

There is a chart  $(\tilde{U}, G, \varphi) \in \alpha$  such that  $x \in \varphi(\tilde{U})$  and embeddings  $\lambda_i: (\tilde{U}, G, \varphi) \rightarrow (\tilde{U}_i, G_i, \varphi_i)$ , for  $i = 1, 2$ . Let  $\tilde{x} \in \tilde{U}$  be such that  $\varphi(\tilde{x}) = x$ . We may assume that  $G_{\tilde{x}} = G$ . Let  $\theta_i: G \rightarrow G_i$  denote the injective homomorphisms associated with the embeddings  $\lambda_i$ . Then  $\tilde{f}_i \circ \lambda_i: \tilde{U} \rightarrow \tilde{V}_i$ , is a  $\theta_i$ -equivariant embedding and  $(\tilde{f}_i(\lambda_i(\tilde{U})), G, \psi_i|)$  is an orbifold chart of  $Y$ , for  $i = 1, 2$ . Now,  $f(x) = f(\varphi(\tilde{x})) \in \psi_1(\tilde{f}_1(\lambda_1(\tilde{U}))) \cap \psi_2(\tilde{f}_2(\lambda_2(\tilde{U})))$ . Let  $\tilde{z}_i = \tilde{f}_i(\lambda_i(\tilde{x}))$ , for  $i = 1, 2$ . There is a chart  $(\tilde{V}, G, \psi)$  of  $Y$  such that  $f(x) \in \psi(\tilde{V})$ , with embeddings  $\mu_i: (\tilde{V}, G, \psi) \rightarrow (\tilde{f}_i(\lambda_i(\tilde{U})), G, \psi_i|)$ . Let  $\tilde{z} \in \tilde{V}$  be such that  $\mu_i(\tilde{z}) = \tilde{z}_i$ . Then  $G_{\tilde{z}} = G$ . The charts  $(\lambda^{-1}(\tilde{f}_1^{-1}(\mu_1(\tilde{V}))), G, \varphi|)$  and  $(\tilde{V}, G, \psi)$ , of  $X$  and  $Y$ , respectively, satisfy the Conditions 2 (a) - (d) of Proposition 2.5. Consequently,  $(\lambda^{-1}(\tilde{f}_1^{-1}(\mu_1(\tilde{V}))), G, \varphi|) \in \alpha_0$  and  $(\tilde{V}, G, \psi) \in f(\alpha_0)$ .  $\square$

### 3 The Proofs

We first recall some well-known facts having to do with *quotient orbifolds*, for more details see [3], Section 3. Let  $G$  be a Lie group and let  $M$  and  $N$  be real analytic manifolds. Assume  $G$  acts on  $M$  and  $N$ , respectively, by proper, effective, almost free, real analytic actions. (An action of  $G$  on  $M$  is proper if the map  $G \times M \rightarrow M \times M$ ,  $(g, x) \mapsto (x, gx)$ , is proper. It is almost free if all the isotropy subgroups are finite.) Then the orbit space  $M/G$  is a reduced real analytic orbifold. The orbifold charts of  $M/G$  are the triples  $(N_x, G_x, \pi_x)$ , where  $x \in M$ ,  $N_x$  is a *linear slice* at  $x$ ,  $G_x$  is the isotropy subgroup at  $x$  and  $\pi_x: N_x \rightarrow N_x/G_x \cong (GN_x)/G$  is the natural projection. Every  $G$ -equivariant  $C^r$ -diffeomorphism  $f: M \rightarrow N$ ,  $1 \leq r \leq \omega$ , induces a  $C^r$ -diffeomorphism  $\tilde{f}: M/G \rightarrow N/G$ .

We next recall the definition and some properties of the *frame bundle* of a reduced real analytic orbifold. For proofs and details, see [4], pp. 42–43. In [4], the frame bundle is constructed for reduced smooth orbifolds, but the same construction goes through in the real analytic case.

Let  $X$  be a reduced real analytic orbifold of dimension  $n$  and let

$$\mathcal{U} = \{(\tilde{U}_i, G_i, \varphi_i)\}_{i \in I}$$

be the maximal orbifold atlas of  $X$ . We first construct the frame bundle  $\text{Fr}(\tilde{U}_i)$  for every chart  $\tilde{U}_i$ . The action of  $G_i$  on  $\tilde{U}_i$  lifts to a left action on  $\text{Fr}(\tilde{U}_i)$ :  $g(x, B) = (gx, (dg)_x \circ B)$ , for every  $(x, B) \in \text{Fr}(\tilde{U}_i)$ . This action is free and it commutes with the right action of the general linear group  $\text{GL}_n(\mathbb{R})$ . In particular,  $\text{Fr}(\tilde{U}_i)/G_i$  is a real analytic manifold on which  $\text{GL}_n(\mathbb{R})$  acts from the right by a proper, effective, almost free, real analytic action, and we may identify  $\text{Fr}(\tilde{U}_i)/G_i$  with the twisted product  $\tilde{U}_i \times_{G_i} \text{GL}_n(\mathbb{R})$ . Let  $p_i: \text{Fr}(\tilde{U}_i)/G_i \rightarrow \tilde{U}_i/G_i$  denote the natural projection.

Let  $\lambda: (\tilde{U}_i, G_i, \varphi_i) \rightarrow (\tilde{U}_j, G_j, \varphi_j)$  be a real analytic embedding between orbifold charts. Let  $\theta: G_i \rightarrow G_j$  be the homomorphism associated with  $\lambda$ . Then  $\lambda$ , together with the differential  $d\lambda$ , induces an embedding  $\tilde{\lambda} = (\lambda, d\lambda): \text{Fr}(\tilde{U}_i) \rightarrow \text{Fr}(\tilde{U}_j)$ . Since  $\lambda$  is  $\theta$ -equivariant, it follows that this embedding factors as

$$\lambda_*: \text{Fr}(\tilde{U}_i)/G_i \rightarrow \text{Fr}(\tilde{U}_j)/G_j.$$

The map  $\lambda_*$  is a real analytic open embedding, it commutes with the action of  $\text{GL}_n(\mathbb{R})$  and  $p_j \circ \lambda_* = p_i$ .

The manifolds  $\text{Fr}(\tilde{U}_i)/G_i$ , for all  $i \in I$ , together with the real analytic embeddings  $\lambda_*$  induced by all the embeddings  $\lambda$  between the orbifold charts, form a filtered direct system. The *frame bundle*  $\text{Fr}(X)$  of the real analytic orbifold  $X$  is defined to be the colimit of this system,

$$\text{Fr}(X) = \varinjlim \{\text{Fr}(\tilde{U}_i)/G_i, \lambda_*\}.$$

Then  $\text{Fr}(X)$  is a real analytic manifold, each  $\text{Fr}(\tilde{U}_i)/G_i$  is canonically embedded into  $\text{Fr}(X)$  as an open submanifold and the maps  $p_i$  induce an open map  $p: \text{Fr}(X) \rightarrow X$ . The following theorem lists some of the properties of  $\text{Fr}(X)$ :

**Theorem 3.1** *Let  $X$  be a reduced real analytic orbifold of dimension  $n$  and let  $\text{Fr}(X)$  be the frame bundle of  $X$ . Then  $\text{Fr}(X)$  is a real analytic manifold. The general linear group  $\text{GL}_n(\mathbb{R})$  acts on  $\text{Fr}(X)$  by a proper, effective, almost free, real analytic action. The quotient orbifold  $\text{Fr}(X)/\text{GL}_n(\mathbb{R})$  is real analytically diffeomorphic to  $X$ .*

**Proof** The other parts of the claim except properness are explained in [4], pp. 42 - 43, in the case of smooth orbifolds. The real analytic case is similar. Since the action of  $\text{GL}_n(\mathbb{R})$  on  $\text{Fr}(X)$  is obviously Cartan (Definition 1.1.2 in [6]) and since  $\text{Fr}(X)/\text{GL}_n(\mathbb{R})$  is regular, it follows from Proposition 1.2.5 in [6] that the action is proper.  $\square$

**Lemma 3.2** *Let  $X$  and  $Y$  be  $n$ -dimensional reduced real analytic orbifolds, and let  $f: X \rightarrow Y$  be a  $C^r$ -diffeomorphism,  $2 \leq r \leq \omega$ . Then  $f$  induces a  $\text{GL}_n(\mathbb{R})$ -equivariant*

$C^{r-1}$ -diffeomorphism  $\hat{f}: \text{Fr}(X) \rightarrow \text{Fr}(Y)$ , and the diagram

$$\begin{array}{ccc} \text{Fr}(X) & \xrightarrow{\hat{f}} & \text{Fr}(Y) \\ \downarrow p_X & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

**Proof** Let  $\alpha$  and  $\beta$  be the real analytic differential structures of  $X$  and  $Y$ , respectively. By Theorem 2.6,  $\alpha$  and  $\beta$  have refinements  $\alpha_0$  and  $f(\alpha_0)$ , respectively, with the property that for every chart  $(\tilde{U}_i, G_i, \varphi_i)$  in  $\alpha_0$  there is a chart  $(\tilde{V}_i, G_i, \psi_i)$  in  $f(\alpha_0)$  such that the charts  $(\tilde{U}_i, G_i, \varphi_i)$  and  $(\tilde{V}_i, G_i, \psi_i)$  satisfy the Conditions 2 (a) - (d) of Proposition 2.5. Denote the restrictions of  $f$  to  $\varphi_i(\tilde{U}_i)$  by  $f_i: \varphi_i(\tilde{U}_i) \rightarrow \psi_i(\tilde{V}_i)$ . Then  $f_i$  has a lift  $\tilde{f}_i: \tilde{U}_i \rightarrow \tilde{V}_i$ , and  $\tilde{f}_i$  is a  $G_i$ -equivariant  $C^r$ -diffeomorphism. The  $C^{r-1}$ -maps  $(\tilde{f}_i, d\tilde{f}_i): \tilde{U}_i \times \text{GL}_n(\mathbb{R}) \rightarrow \tilde{V}_i \times \text{GL}_n(\mathbb{R})$  induce  $\text{GL}_n(\mathbb{R})$ -equivariant  $C^{r-1}$ -diffeomorphisms

$$\tilde{f}_i: \text{Fr}(\tilde{U}_i)/G_i \cong \tilde{U}_i \times_{G_i} \text{GL}_n(\mathbb{R}) \rightarrow \tilde{V}_i \times_{G_i} \text{GL}_n(\mathbb{R}) \cong \text{Fr}(\tilde{V}_i)/G_i.$$

Let  $\lambda: \tilde{U}_i \rightarrow \tilde{U}_j$  and  $\mu: \tilde{V}_i \rightarrow \tilde{V}_j$  be embeddings. Then  $\tilde{f}_j^{-1} \circ \mu \circ \tilde{f}_i: \tilde{U}_i \rightarrow \tilde{U}_j$  is also an embedding. By Proposition A.1 in [5], there is a unique  $g \in G_i$  such that  $\lambda = g \circ \tilde{f}_j^{-1} \circ \mu \circ \tilde{f}_i$ . Therefore,  $\tilde{f}_j \circ \lambda = g \circ \mu \circ \tilde{f}_i$ . It follows that the maps  $\tilde{f}_i$  commute with the embeddings

$$\lambda_*: \text{Fr}(\tilde{U}_i)/G_i \rightarrow \text{Fr}(\tilde{U}_j)/G_j \text{ and } \mu_*: \text{Fr}(\tilde{V}_i)/G_i \rightarrow \text{Fr}(\tilde{V}_j)/G_j$$

that are used to define the frame bundles  $\text{Fr}(X)$  and  $\text{Fr}(Y)$ , respectively. Thus they induce a  $\text{GL}_n(\mathbb{R})$ -equivariant  $C^{r-1}$ -diffeomorphism  $\hat{f}: \text{Fr}(X) \rightarrow \text{Fr}(Y)$ .  $\square$

Notice that in Theorem 1.1 we assume that  $2 \leq r$  instead of  $1 \leq r$ . The reason for doing so is that the proof of Theorem 1.1 uses Lemma 3.2.

*Proof of Theorem 1.1.* Let  $X$  and  $Y$  be reduced  $C^r$ -orbifolds,  $2 \leq r \leq \omega$ , and let  $f: X \rightarrow Y$  be a  $C^2$ -diffeomorphism. According to Theorem 8.4 in [2], there are real analytic orbifolds  $X^\omega$  and  $Y^\omega$  and  $C^r$ -diffeomorphisms  $f_1: X \rightarrow X^\omega$  and  $f_2: Y \rightarrow Y^\omega$ . By Theorem 3.1, there are real analytic diffeomorphisms  $h_1: X^\omega \rightarrow \text{Fr}(X^\omega)/\text{GL}_n(\mathbb{R})$  and  $h_2: Y^\omega \rightarrow \text{Fr}(Y^\omega)/\text{GL}_n(\mathbb{R})$ . Let  $g = f_2 \circ f \circ f_1^{-1}: X^\omega \rightarrow Y^\omega$ . Then  $g$  is a  $C^2$ -diffeomorphism, and by Lemma 3.2 it induces a  $\text{GL}_n(\mathbb{R})$ -equivariant  $C^1$ -diffeomorphism  $\hat{g}: \text{Fr}(X^\omega) \rightarrow \text{Fr}(Y^\omega)$ . Now,  $\hat{g}$  induces a  $C^1$ -diffeomorphism  $\tilde{g}: \text{Fr}(X^\omega)/\text{GL}_n(\mathbb{R}) \rightarrow \text{Fr}(Y^\omega)/\text{GL}_n(\mathbb{R})$  and the diagram

$$\begin{array}{ccc}
\mathrm{Fr}(X^\omega) & \xrightarrow{\hat{g}} & \mathrm{Fr}(Y^\omega) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
\mathrm{Fr}(X^\omega)/\mathrm{GL}_n(\mathbb{R}) & \xrightarrow{\tilde{g}} & \mathrm{Fr}(Y^\omega)/\mathrm{GL}_n(\mathbb{R}) \\
\downarrow h_1^{-1} & & \downarrow h_2^{-1} \\
X^\omega & \xrightarrow{g} & Y^\omega \\
\downarrow f_1^{-1} & & \downarrow f_2^{-1} \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. By Corollary IIa in [1], there is a  $\mathrm{GL}_n(\mathbb{R})$ -equivariant real analytic diffeomorphism  $\hat{h}: \mathrm{Fr}(X^\omega) \rightarrow \mathrm{Fr}(Y^\omega)$ . Then  $\hat{h}$  induces a real analytic diffeomorphism  $\tilde{h}: \mathrm{Fr}(X^\omega)/\mathrm{GL}_n(\mathbb{R}) \rightarrow \mathrm{Fr}(Y^\omega)/\mathrm{GL}_n(\mathbb{R})$ . It follows that  $h = h_2^{-1} \circ \tilde{h} \circ h_1: X^\omega \rightarrow Y^\omega$  is a real analytic diffeomorphism. Therefore,  $h^r = f_2^{-1} \circ h \circ f_1: X \rightarrow Y$  is a  $C^r$ -diffeomorphism.  $\square$

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